Error estimation in the Monte Carlo method for calculating solar radiances; Illustration with cloud models based on Landsat measurements

A. Marshak (UMBC), A. B. Davis (LANL), L. Oreopoulos (UMBC), T. Várnai (UMBC)

In this paper, we first highlight the general concept of Monte Carlo (MC) error estimation based on well-known formulae from the theory of large numbers (e.g., Papoulis, 1965; p. 149). Next, we briefly describe the methods of local estimates and maximum cross sections used for calculating zenith and nadir radiances (Marchuk, 1980). Finally, simulated radiances with associated errors are illustrated and discussed using the Intercomparison of 3D Radiation Codes (I3RC) Case 3 cloud model based on Landsat measurements.

General concept of MC error estimate

Let us assume that we obtained N independent realizations ξ_i (i=1, …, N) of a random value ξ with finite mathematical expectation $E\xi$ and variance $D\xi$. If N is large enough, then the average of ξ_is has a normal distribution and the inequality

$$|E\xi - \frac{1}{N}\sum_{i=1}^{N} \xi_i| \le c_\beta \sqrt{D\xi/N} \tag{1}$$

is valid at a given confidence level β which defines the constant c_β. For example, c_β=0.67 if β = 0.5, c_β=1.0 if β = 0.68 (the “1-sigma” value), c_β=1.96 if β = 0.95, and c_β=3 if β = 0.997 (the famous rule of “3 sigmas”). More precisely,

$$\mathrm{Prob}\left\{|E\xi - \frac{1}{N}\sum_{i=1}^{N} \xi_i| < c\sqrt{\frac{D\xi}{N}}\right\} \approx \Phi(c) = \frac{2}{\sqrt{2\pi}}\int_0^c \exp(-t^2/2)dt \tag{2}$$

and c_β is the solution of the equation,

$$\Phi(c) = \beta. \tag{3}$$

The variance $D\xi$ can be estimated as

$$D\xi \approx \frac{N}{N-1}\left\{\frac{1}{N}\sum_{i=1}^{N} \xi_i^2 - [\frac{1}{N}\sum_{i=1}^{N} \xi_i]^2\right\}. \tag{4}$$

To conclude, based on (1), $\sqrt{D\xi/N}$ is an estimate of MC uncertainties with a confidence level β = 0.68. This means that after N trials, with 68% probability, the MC estimate, $(1/N)\sum_{i=1}^{N} \xi_i$, will have an error smaller than $\sqrt{D\xi/N}$ where variance $D\xi$ is estimated using Eq. (4).

Calculations of nadir and zenith radiances as a local estimate

The upward or downward radiances I_j for each cell S_j on a horizontal grid, can be estimated by the flux of radiant energy across:

- the upper boundary of S_j (at $z = h$) in the zenith direction (Ω_+), or
- the lower boundary of S_j (at $z = 0$) in the nadir direction (Ω_-),

$$I_{\boldsymbol{j}}(\Omega_\pm) = \int_{S_{\boldsymbol{j}}} I(\boldsymbol{x};\Omega_\pm)\,\mathrm{d}\boldsymbol{x} \Big/ \int_{S_{\boldsymbol{j}}} \mathrm{d}\boldsymbol{x} = E[\xi_{\boldsymbol{j}}(\Omega_\pm)] \approx \frac{1}{N}\sum_{i=1}^{N} \xi_{\boldsymbol{j}_i}(\Omega_\pm). \qquad (5)$$

Here $\xi_{\boldsymbol{j}_i}$ ($i = 1,\ldots,N$) are N independent realizations (photon trajectories) of a random function $\xi_{\boldsymbol{j}}$. For each realization (a trajectory), the random value $\xi_{\boldsymbol{j}}(\Omega_\pm)$ is the contribution to the grid-point $S_{\boldsymbol{j}}$ into the direction $\Omega_\pm$ from all orders of scattering:

$$\xi_{\boldsymbol{j}}(\Omega_\pm) = \sum_{k=1}^{m} \varpi_0^{k} P(\Omega_k \bullet \Omega_\pm)\chi_{\boldsymbol{j}}(k)\begin{cases} \exp[-\sigma_{\boldsymbol{j}}(h - z_k)], & \text{zenith } (+) \\ \exp[-\sigma_{\boldsymbol{j}} z_k], & \text{nadir } (-) \end{cases}, \qquad (6)$$

m is the (random) last scattering order of the photon trajectory, $P(\Omega \bullet \Omega')$ is the scattering phase function; ϖ_0 is the single scattering albedo, $\sigma_{\boldsymbol{j}}$ is the extinction of the grid-point $S_{\boldsymbol{j}}$, $\boldsymbol{r}_k = (x_k, y_k, z_k)$ are the coordinates of the point of photon's kth scattering, Ω_k is its direction of propagation before this scattering event, and finally, $\chi_{\boldsymbol{j}}(k)$ indicates whether the photon was in cell S_j or not at its kth scattering:

$$\chi_{\boldsymbol{j}}(k) = \begin{cases} 1, \text{ if } (x_k y_k) \in S_{\boldsymbol{j}} \\ 0, \text{ otherwise} \end{cases}. \qquad (7)$$

Photon trajectories, points $\boldsymbol{r}_k$, are calculated using phase function $P(\Omega \bullet \Omega')$ for the direction of travel and the maximum cross-section method with a new (but constant!) extinction

$$\sigma_{\max} = \max_{\boldsymbol{r}} [\sigma(\boldsymbol{r})]. \qquad (8)$$

for a photon's free path. Hence, the length of photon step l is simulated using probability density function

$$p(x) = \sigma_{\max} \exp(-\sigma_{\max} x); \qquad (9)$$

thus

$$l = -\ln \alpha / \sigma_{\max}, \qquad (10)$$

where α is a random number uniformly distributed on (0,1].

The uncertainties in simulation (5) are estimated similar to those described in (1)-(3) where variance $D[\xi_{\boldsymbol{j}}(\Omega_\pm)]$ is calculated as described in Eq. (4). In other words, in addition to the first moment we also accumulate the second moment. In practice, it is more efficient to calculate the second moment not after each photon but after a package of photons. Some ideas on "packaging" photons can be found in Marchuk et al. (1980).

Maximum cross-section method

The idea of the maximum cross-section method is simple (Marchuk et al., 1980, p. 9): the traditional 3D radiative transfer equation is identical to the following equation,

$$\Omega \bullet \nabla I(\boldsymbol{r};\Omega) + \sigma_{\max} I(\boldsymbol{r};\Omega) = \sigma_{\max} \int_{4\pi} \left[\frac{\sigma(\boldsymbol{r})}{\sigma_{\max}} \varpi_0 P(\Omega \bullet \Omega') + \left(1 - \frac{\sigma(\boldsymbol{r})}{\sigma_{\max}}\right)\delta(\Omega - \Omega')\right] I(\boldsymbol{r};\Omega')\,\mathrm{d}\Omega' \qquad (11)$$

where δ is the Dirac delta function. The above equation can be interpreted as the transport equation with constant extinction and a modified phase function equal to

$$\begin{cases} \varpi_0 P(\Omega \bullet \Omega'), & \text{with probability } \sigma(\boldsymbol{r})/\sigma_{\max} \quad \text{(this is a "physical" scattering)} \\ \delta(\Omega - \Omega'), & \text{otherwise} \quad \text{(this is a "mathematical" scattering)} \end{cases}. \qquad (12)$$

Radiance uncertainties for the cloud fields retrieved from Landsat data.

In this section we illustrate the above theory with examples from the Case 3 of the I3RC (see http://climate.gsfc.nasa.gov/I3RC/). This case represents a marine boundary-layer cloud retrieved from Landsat.

The MC code used in the I3RC had been developed at Goddard's Climate and Radiation Branch; it was originated by G. Titov and A. Marshak and then further modified by all co-authors. The code is based on the MC solution of the monochromatic 3D radiative transfer equation for a given 3D extinction field and any surface reflective properties. The code simultaneously calculates downward/upward fluxes and cloud absorption, as well as radiances in any given direction using the methods of maximum cross-section and local estimate described above. For the Landsat case (128x128 pixels or 3.8x3.8 km), the code was run on SGI Origin 200 machine (180 MHz, SPECfp_base 95 = 14.5). The processing rate for the simulation was 4-5 10^3 photons per 1 sec or 15-20 10^6 photons per 1 h.

Figure 1 shows zenith (downward) radiances calculated with 10^8 and 10^9 photons, respectively, for solar zenith angle (SZA) of 60º. It is clearly seen that the lower image (with 10^9 photons) has much smaller noise than the upper one. The majority of pixels in the lower image have an error around 1%. The errors at the boundary are much larger and often exceed 5 and 10%. (This will be discussed in more detail in Fig. 3.) What is of interest is that, if we did not have colorbars for the rightmost panels, the two images of relative errors would be statistically indistinguishable. Indeed, it follows directly from Eq. (2) that the increase of number of independent realizations by 10 will decrease MC noise by $\sqrt{10} \approx 3.2$. This is approximately [but not exactly, because, in general, $D\xi = D\xi(N)$ from Eq. (4)] what we see if we compare the two colorbars.

Figure 2 illustrates nadir (upward) radiances for the same SZA = 60º. In addition to relative errors (two middle panels), we show absolute errors for both 10^8 and 10^9 photons. We see that while the relative errors for most pixels are slightly less that 1% (for 10^9 photons), the absolute errors are about 0.005. (As explained above, this means that with about 70% probability, the second decimal of our pixel-by-pixel nadir radiances results is either correct or differs from the true value by 1.) Note that, as in Fig. 1, both upper and lower middle and the right panels are statistically equal. They can be only distinguished by looking at the colorbars.

Finally, Fig. 3 details an area near the boundary of the scene showing nadir (upper middle and right) and zenith (lower middle and right panels) radiances calculated with 10^9 photons. In addition, we plotted optical (upper left) and geometrical thicknesses (lower left panel). Both images show a 1.2x1.2 km area at the lowest left corner of the original image. We see that the largest MC errors happen at the very edge of the image where the total optical thickness is 0.1-0.2 and cloud geometrical thickness is about 0.5 km. Hence the extinction of these pixels is very small and very few interactions occur there. As a result, a contribution to the radiances from these pixels is rare and the variance is huge. If the radiance for these pixels has a significant value, we can calculate the contribution from the first order of scattering analytically (or numerically) and the rest by MC. This will definitely lead to the reduction of variance. For more details, see E. Kasyanov's abstract (this volume).

Bibliography

Marchuk, G., G. Mikhailov, M. Nazaraliev, R. Darbinjan, B. Kargin, and B. Elepov, 1980: *The Monte Carlo Methods in Atmospheric Optics*, 208 pp., Springer-Verlag, New-York (NY).

Papoulis, A., 1965: *Probability, Random Variables, and Stochastic Processes*. McGraw-Hill, New York (NY), xi+583pp.

Fig. 1 Zenith radiances (Id) calculated with 10^8 and 10^9 photons, respectively, and their relative errors. Solar zenith angle is 60^0. The leftmost column shows 2D fields of zenith radiances. The middle column shows horizontal cuts along the row #101. The rightmost column shows relative errors in %.

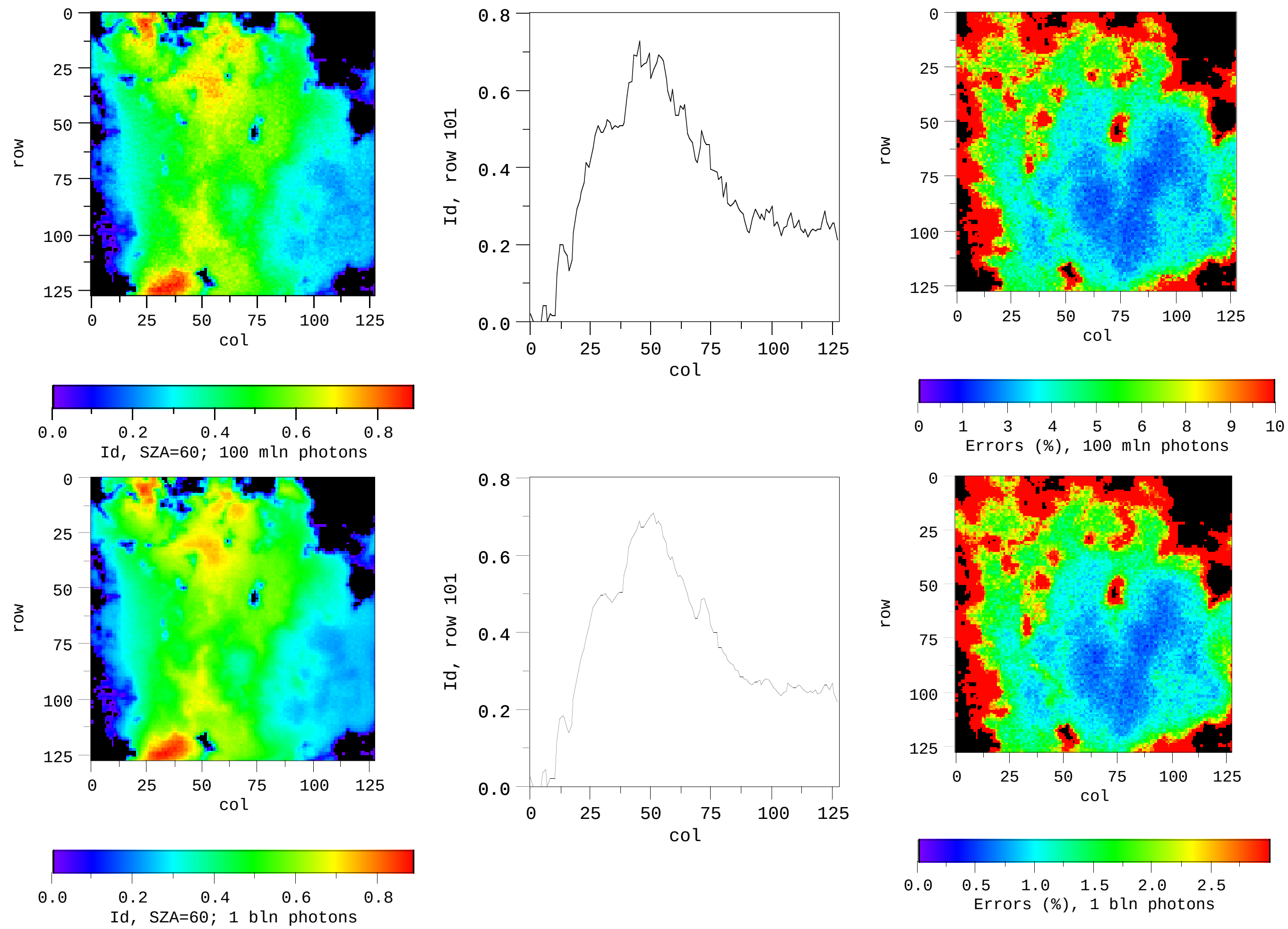

Fig. 2 The same as in Fig. 1 but for nadir radiances (Iu). The leftmost column shows 2D fields of nadir radiances. The middle column shows relative errors in % while the rightmost column shows absolute errors.

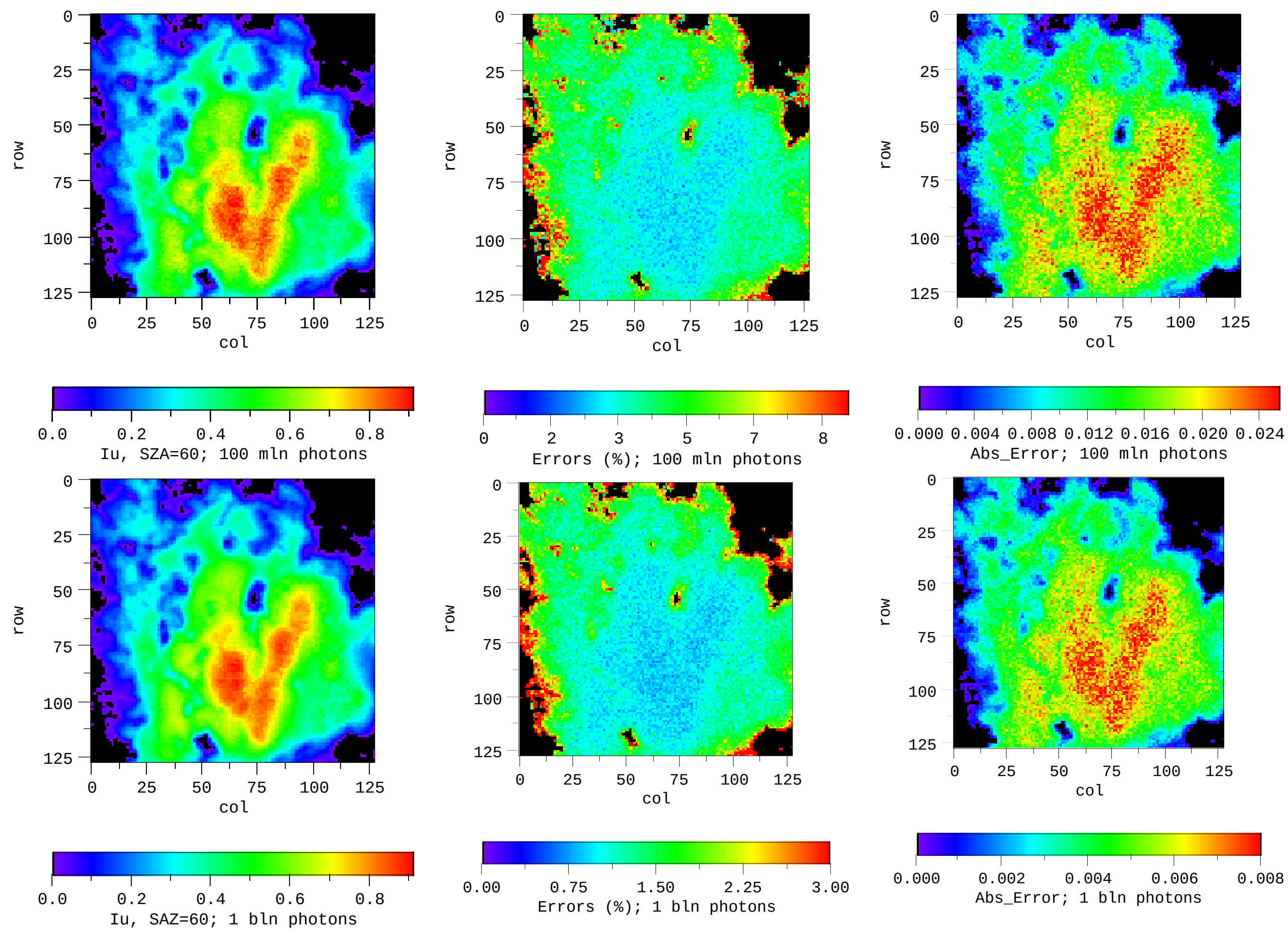

Fig. 3 Fragments of zenith and nadir radiances calculated with 10^9 photons. Solar zenith angle is 60^0. The leftmost column shows optical (upper panel) and geometrical (lower panel) thicknesses of a cloud field. The middle column shows zenith and nadir radiances as fragments from those plotted in Figs. 1 and 2, respectively. The rightmost column shows relative errors in %.